ON REPRESENTATIONS OF CLASSICAL GROUPS OVER PRINCIPAL IDEAL LOCAL RINGS OF LENGTH TWO

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ABSTRACT. We study the complex irreducible representations of special linear, symplectic, orthogonal and unitary groups over principal ideal local rings of length two. We construct a canonical correspondence between the irreducible representations of all such groups that preserves dimensions. The case for general linear groups has already been proved by author.

1. Introduction

Let F be a non-Archimedean local field with ring of integers $\mathcal{O}$. Let $\wp$ be the unique maximal ideal of $\mathcal{O}$ and π be a fixed uniformizer of $\wp$. Assume that the residue field $\mathcal{O}/\wp$ has odd characteristic p. We denote by $\mathcal{O}_\ell$ the reduction of $\mathcal{O}$ modulo $\wp^\ell$, i.e., $\mathcal{O}_\ell = \mathcal{O}/\wp^\ell$. Therefore $\mathcal{O}_1$ will denote the residue field of $\mathcal{O}$.

The representation theory of classical groups over the rings $\mathcal{O}$ and the finite rings $\mathcal{O}_\ell$ has attracted attention of many mathematcians. See Singla [7] for the history of this problem for the General and Special linear groups over the rings $\mathcal{O}$ and their finite quotients $\mathcal{O}_\ell$. In the direction of the classical groups, Lusztig [5] constructed several irreducible representations of reductive groups over finite rings and Jaikin-Zapirain [4] looked at the problem of constructing irreducible representations of compact pro-p groups. But the knowledge of all irreducible representations of classical groups over $\mathcal{O}$ and the finite rings $\mathcal{O}_\ell$ is still far from complete.

In Singla [7], we gave a method to construct irreducible representations of general linear groups $\mathrm{GL}_n(\mathcal{O}_2)$. In this article, we use it to construct irreducible representations of other classical groups over the rings $\mathcal{O}_2$. The questions of this article are also motivated by a conjecture of Onn [6, Conjecture 1.2], which says that the isomorphism type of the group algebra of automorphism group of finite $\mathcal{O}$-modules depend on the ring of integers only through the cardinality of its residue field. More generally one can ask this question for the other classical groups over $\mathcal{O}_\ell$ as well. In Singla [7], we proved Onn's conjecture for groups $\mathrm{GL}_n(\mathcal{O}_2)$.

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1.1. **Main Results.** The classical groups over ring $\mathcal{O}_2$ are defined as:

(1) Special Linear Group: $\mathrm{SL}_n(\mathcal{O}_2) = \{g \in \mathrm{GL}_n(\mathcal{O}_2) \mid \det(g) = 1\}$.

(2) Symplectic Group: $\mathrm{Sp}_n(\mathcal{O}_2) = \{g \in \mathrm{GL}_{2n}(\mathcal{O}_2) \mid g^{\mathrm{t}} J g = J\}$, where g^{t} denotes transpose of g and $J = \left[\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right]$.

(3) Orthogonal Group: $\mathrm{O}_n(\mathcal{O}_2) = \{g \in \mathrm{GL}_n(\mathcal{O}_2) \mid g^{\mathrm{t}} g = I_n\}$.

(4) Unitary Group: Let $\tilde{F}$ be a degree two unramifield extension of F and σ be the unique nontrivial Galois automorphism of $\tilde{F}$. Let $\tilde{\mathcal{O}}_2$ be the corresponding unramified extension of $\mathcal{O}_2$, then σ restricts to an automorphism of $\tilde{\mathcal{O}}_2$ (denoted again as σ). Applying σ entry wise we obtain an automorphism of $\mathrm{GL}_n(\tilde{\mathcal{O}}_2)$. For any $g \in \mathrm{GL}_n(\tilde{\mathcal{O}}_2)$, let $g^\star = (g^\sigma)^{\mathrm{t}}$. Then the unitary group is defined as
$$\mathrm{U}_n(\mathcal{O}_2) = \{g \in \mathrm{GL}_n(\tilde{\mathcal{O}}_2) \mid g g^\star = I_n\}.$$

We prove that the number and dimensions of irreducible representations of these classical groups over $\mathcal{O}_2$ depend on $\mathcal{O}$ only through the cardinality of residue field. More precisely we prove the following.

Let F and F$'$ be local fields with rings of integers $\mathcal{O}$ and $\mathcal{O}'$, respectively, such that their residue fields are finite and isomorphic (with a fixed isomorphism). Let $\wp$ and $\wp'$ be the maximal ideals of $\mathcal{O}$ and $\mathcal{O}'$ respectively. As described earlier, $\mathcal{O}_2$ and $\mathcal{O}'_2$ denote the rings $\mathcal{O}/\wp^2$ and $\mathcal{O}'/\wp'^2$, respectively. For a ring R, we use $C(R)$ as collective notation for any of the classical groups $\mathrm{SL}_n(R)$, $\mathrm{Sp}_n(R)$, $\mathrm{O}_n(R)$, or $\mathrm{U}_n(R)$ over R.

Theorem 1.1. *There exists a canonical bijection between the irreducible representations of $C(\mathcal{O}_2)$ and those of $C(\mathcal{O}'_2)$, which preserves dimensions.*

By the equivalence between the number of conjugacy classes and distinct irreducible representations, we also obtain that the number of conjugacy classes of the classical groups over $\mathcal{O}_2$ depend on $\mathcal{O}$ only through $|\mathcal{O}_1|$. We remark that very little is known about the conjugacy classes of classical group $C(\mathcal{O}_\ell)$ (See [2, 1, 8]).

2. Proof of Theorem 1.1

First of all, we set up few notations. By character we shall always mean one-dimensional representation. For an abelian group A, the set of its characters is denoted by $\hat{A}$. For any group G, the set of inequivalent irreducible representations

is denoted by Irr$G$. Let

$$\kappa : \mathrm{GL}_n(\mathcal{O}_2) \to \mathrm{GL}_n(\mathcal{O}_1) \text{ and } \bar{\kappa} : C(\mathcal{O}_2) \to C(\mathcal{O}_1)$$

be the natural quotient maps with $K = \ker(\kappa)$ and $L(C) = \ker(\bar{\kappa})$. We shall use the following results of Clifford theory.

Theorem 2.1 (Clifford Theory). *Let G be a finite group and N be a normal subgroup. Let ρ be an irreducible representation of N and $T(\rho) = \{g \in G \mid \rho^g = \rho\}$ be the stabilizer of ρ. Then the following hold*

(1) *If π is an irreducible representation of G such that $\langle \pi|_N, \rho\rangle \neq 0$, then $\pi|_N = e(\oplus_{\rho\in\Omega}\rho)$ where Ω is an orbit of irreducible representations of N under the action of G, and e is a positive integer.*

(2) *Let $A = \{\theta \in \mathrm{Irr}(T(\rho)) \mid \langle \mathrm{Res}_N^{T(\rho)}\theta, \rho\rangle \neq 0\}$ and*
$$B = \{\pi \in \mathrm{Irr}G \mid \langle \mathrm{Res}_N^G \pi, \rho\rangle \neq 0\}.$$
Then
$$\theta \to \mathrm{Ind}_{T(\rho)}^G(\theta)$$
is a bijection of A onto B.

(3) *Let H be a subgroup of G containing N, and suppose that ρ has an extension $\tilde{\rho}$ to H (i.e., $\tilde{\rho}|_N = \rho$). Then the representations $\chi \otimes \tilde{\rho}$ for $\chi \in \mathrm{Irr}(H/N)$ are irreducible, distinct for distinct χ, and*

$$\mathrm{Ind}_N^H(\rho) = \oplus_{\chi\in\mathrm{Irr}(H/N)}\chi \otimes \tilde{\rho}.$$

Let $\widehat{L(C)}$ denote the set of characters of $L(C)$. The group $C(\mathcal{O}_2)$ acts on $\widehat{L(C)}$ by conjugation. That is, if $\alpha \in C(\mathcal{O}_2)$ and $\phi \in \widehat{L(C)}$ then $\phi^\alpha(x) = \phi(\alpha x \alpha^{-1})$ for $x \in L(C)$. For any $\phi \in \widehat{L(C)}$, let $T_C(\phi) = \{\alpha \in C(\mathcal{O}_2) \mid \phi^\alpha = \phi\}$ be the stabilizer of ϕ in $C(\mathcal{O}_2)$.

Proposition 2.2. *For any $\phi \in \widehat{L(C)}$,*

(a) *There exists a canonical character χ_ϕ of $T_C(\phi)$ such that $\chi_\phi|_{L(C)} = \phi$.*
(b) *The group $T_C(\phi)/K$ depends on $\mathcal{O}$ only through $|\mathcal{O}_1|$.*
(c) *The cardinality $|C(\mathcal{O}_2)/T_C(\phi)|$ depends on $\mathcal{O}$ only through $|\mathcal{O}_1|$.*

We postpone the proof of this proposition to § 3 and 4. Assuming this, we complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\mathcal{S}_C$ denote the set of $C(\mathcal{O}_2)$-orbits in $\widehat{L(C)}$ under conjugation. Therefore, by Clifford Theory, there exists a bijection (also canonical by proposition 2.2) between the sets

$$\coprod_{\phi \in \mathcal{S}_C} \{\mathrm{Irr}(T_C(\phi)/K)\} \quad \longleftrightarrow \quad \mathrm{IrrC}(\mathcal{O}_2), \tag{2.1}$$

given by,

$$\delta \mapsto \mathrm{Ind}_{T_C(\phi)}^{C(\mathcal{O}_2)}(\chi_\phi \otimes \tilde{\delta}), \tag{2.2}$$

where $\tilde{\delta}$ is a representation of $T_C(\phi)$ obtained by composing δ with the natural projection map $T_C(\phi) \to T_C(\phi)/K$. Since the left side of (2.1) depends on $\mathcal{O}$ only through the cadinality of $\mathcal{O}_1$, theorem follows.

□

3. The groups $\mathrm{O}_n(\mathcal{O}_2)$, $\mathrm{Sp}_n(\mathcal{O}_2)$, $\mathrm{U}_n(\mathcal{O}_2)$, and $\mathrm{SL}_n(\mathcal{O}_2)(p \nmid n)$

Fix a nontrivial additive character $\psi : \mathcal{O}_1 \to \mathbf{C}^\times$. For each $A \in M_n(\mathcal{O}_1)$ the character $\psi_A : K \to \mathbf{C}^\times$ is defined by

$$\psi_A(I + \pi X) = \psi(\mathrm{Tr}(AX)).$$

The bilinear form $\langle .,. \rangle : M_n(\mathcal{O}_1) \times M_n(\mathcal{O}_1) \to \mathcal{O}_1$, defined by $(A, B) \mapsto \mathrm{Tr}(AB)$ is non-degenerate therefore the assignment $A \mapsto \psi_A$ defines an isomorphism $M_n(\mathcal{O}_1) \cong \hat{K}$. Let $T_G(\psi_A) = \{g \in \mathrm{GL}_n(\mathcal{O}_2) \mid \psi_A^g = \psi_A\}$ (in place of $T(\psi_A)$ of Singla [7] to remove any ambiguity) denote the stabilizer of ψ_A in $\mathrm{GL}_n(\mathcal{O}_2)$. In this section we prove Proposition 2.2 for Orthogonal, Unitary, Symplectic and Special linear groups($p \nmid n$). For this section $C(\mathcal{O}_2)$ denotes either $\mathrm{O}_n(\mathcal{O}_2)$, $\mathrm{U}_n(\mathcal{O}_2)$, $\mathrm{Sp}_n(\mathcal{O}_2)$ or $\mathrm{SL}_n(\mathcal{O}_2)(p \nmid n)$. For any classical group $C(\mathcal{O}_2)$, let M_C be the subgroup of $M_n(\mathcal{O}_1)$ such that $X \mapsto I + \pi X$ defines an isomorphism $M_C \cong L(C)$. By restricting $\langle .,. \rangle$ to M_C, we obtain a bilinear form on M_C as well. Observe that if this restriction is non-degenerate then,

(1) The map $A \mapsto \psi_A|_{L(C)}$ defines an isomorphism, $M_c \cong \widehat{L(C)}$.
(2) $T_C(\psi_A|_{L(C)}) = T_G(\psi_A) \cap \mathbf{C}(\mathcal{O}_2)$.

We recall the following result of Singla [7]

Proposition 3.1. *For any $\phi \in \hat{K}$, there exists a canonical character χ_ϕ^G of $T_G(\phi)$ such that $\chi_\phi^G|_K = \phi$ (such a character χ_ϕ^G is called an extension of ϕ).*

Proof. For proof see Proposition 2.2 and Section 5 of Singla [7]. □

Let $\phi = \psi_A|_{L(C)}$ for some $A \in M_C$, then part (a) of Proposition 2.2 follows by taking $\chi_\phi = \chi_\phi^G|_{T_C(\phi)}$. Parts (b) and (c) of Proposition 2.2 follow by the following facts:

(1) $T_C(\phi)/L(C) \cong Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A) \cap C(\mathcal{O}_1)$.
(2) $|C(\mathcal{O}_2)/T_C(\phi)| = |C(\mathcal{O}_1)/(Z_{\mathrm{GL}_n(\mathcal{O}_1)}(A) \cap C(\mathcal{O}_1))|$.

These follow easily by definitions of $T_C(\phi)$, $L(C)$ and Corollary 5.3 of Singla [7].

We shall show that the bilinear form $\langle ., . \rangle|_{M_C}$ is non-degenerate for Symplectic, Unitary, Orthogonal, and Special linear($p \nmid n$) groups. Further in § 4 we show that it is not true for Special linear group with $p \mid n$, which makes this case bit harder. We shall use a completely different method to solve $\mathrm{SL}_n(\mathcal{O}_2)(p \mid n)$.

3.1. $\mathbf{O}_n(\mathcal{O}_2)$**.** The kernel $L(O)$ of the natural projection map $\mathrm{O}_n(\mathcal{O}_2) \to \mathrm{O}_n(\mathcal{O}_1)$ is isomorphic to

$$M_O = \{X \in \mathrm{M}_n(\mathcal{O}_1) \mid X + X^{\mathrm{t}} = 0\}$$

by $I + \pi X \mapsto X$. For any $A \in M_O$, since $\langle ., . \rangle$ is non-degenerate on $\mathrm{M}_n(\mathcal{O}_1)$, there exists $Y \in \mathrm{M}_n(\mathcal{O}_1)$ such that $\mathrm{Tr}(AY) \neq 0$. Take $B = Y - Y^{\mathrm{t}} \in M_O$. By using additivity and $\mathrm{Tr}(Z) = \mathrm{Tr}(Z^{\mathrm{t}})$ for any $Z \in \mathrm{M}_n(\mathcal{O}_1)$, we obtain

$$\mathrm{Tr}(AB) = \mathrm{Tr}(A(Y - Y^{\mathrm{t}})) = \mathrm{Tr}(AY + A^{\mathrm{t}}Y^{\mathrm{t}}) = 2\mathrm{Tr}(AY) \neq 0.$$

Here we have also used the fact that the residue field is of odd characteristic. Hence $\langle ., . \rangle|_{M_O}$ is non-degenerate.

3.2. $\mathbf{U}_n(\mathcal{O}_2)$**.** The kernel $L(U)$ of the natural projection map $\mathrm{U}_n(\mathcal{O}_2) \to \mathrm{U}_n(\mathcal{O}_1)$ is isomorphic to the set

$$M_U = \{X \in \mathrm{M}_n(\mathcal{O}_1) \mid X + X^\star = 0\}$$

by $I + \pi X \mapsto X$. The rest of the argument for this follows similar to orthogonal group case, by replacing $Y - Y^{\mathrm{t}}$ with $Y - Y^\star$.

3.3. $\mathbf{Sp}_n(\mathcal{O}_2)$**.** The kernel $L(\mathrm{Sp})$ of the natural projection map $\mathrm{Sp}_n(\mathcal{O}_2) \to \mathrm{Sp}_n(\mathcal{O}_1)$ consists of matrices $I + \pi X$, $X \in M_{2n}(\mathcal{O}_2)$ such that

$$(I + \pi X)^{\mathrm{t}} J (I + \pi X) = J, \text{ where } J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Let

$$M_{\mathrm{Sp}} = \{X \in \mathrm{M}_{2n}(\mathcal{O}_1) \mid X^{\mathrm{t}} J + JX = 0\}$$

then $X \mapsto I + \pi X$ is easily seen to give an isomorphism between M_{Sp} and $L(\mathrm{Sp})$. Also observe that,

$$M_{\mathrm{Sp}} = \{\begin{bmatrix} U & V \\ W & -U^{\mathrm{t}} \end{bmatrix} \mid U, V, W \in \mathrm{M}_n(\mathcal{O}_1), V = -V^{\mathrm{t}}, W = W^{\mathrm{t}}\}.$$

For any $A = \begin{bmatrix} U & V \\ W & -U^{\mathrm{t}} \end{bmatrix}$, $X = \begin{bmatrix} U' & V' \\ W' & -U'^{\mathrm{t}} \end{bmatrix} \in M_{\mathrm{Sp}}$,

$$\mathrm{Tr}(AX) = 2\mathrm{Tr}(UU') + \mathrm{Tr}(VW') + \mathrm{Tr}(WV').$$

By using the fact that $(A, B) \mapsto \mathrm{Tr}(AB)$ is a non-degenerate bilinear form on the set of matrices, symmetric and skew-symmetric matrices over $\mathcal{O}_1$. We obtain that for any nonzero $A \in M_{\mathrm{Sp}}$ there exists $X \in M_{\mathrm{Sp}}$ such that $\mathrm{Tr}(AX) \neq 0$. Therefore $\langle ., . \rangle|_{M_{\mathrm{Sp}}}$ is non-degenerate.

3.4. **$\mathbf{SL}_n(\mathcal{O}_2)$, $p \nmid n$:** Let $L(SL)$ denote the kernel of the natural projection map $\mathrm{SL}_n(\mathcal{O}_2) \to \mathrm{SL}_n(\mathcal{O}_1)$. Identify the set $L(\mathrm{SL})$ with $M_{\mathrm{SL}} = \{X \in \mathrm{M}_n(\mathcal{O}_1) \mid \mathrm{Tr}(X) = 0\}$, by $I + \pi X \mapsto X$. We prove that the bilinear form $(A, B) \to \mathrm{Tr}(AB)$ is non-degenerate on M_{SL}.

For any non-diagonal matrix $A = (a_{ij}) \in \mathrm{sl}_n(\mathcal{O}_1)$, there exists pair (i_0, j_0) such that $i_0 \neq j_0$ and $a_{i_0 j_0} \neq 0$. Take $B = (b_{ij}) \in M_{\mathrm{SL}}$ such that $b_{j_0 i_0} = 1$ and zeros everywhere else. Then $\mathrm{Tr}(AB) \neq 0$. On the other hand if $A = (a_{ij}) \in M_{\mathrm{SL}}$ is a non-zero diagonal matrix, then $p \nmid n$ implies there exists $i_0 \neq j_0$ such that $a_{i_0 i_0} \neq a_{j_0 j_0}$. Then by taking $B = (b_{ij}) \in M_{\mathrm{SL}}$ to be the matrix satisfying $b_{i_0 i_0} = 1$, $b_{j_0 j_0} = -1$ and $b_{ij} = 0$ for all $(i, j) \notin \{(i_0, i_0), (j_0, j_0)\}$, we obtain that $\mathrm{Tr}(AB) \neq 0$. This proves the assertion.

4. Special Linear Group $\mathrm{SL}_n(\mathcal{O}_2)$, $p \mid n$

Let $L(SL)$ denote the kernel of the natural projection map $\mathrm{SL}_n(\mathcal{O}_2) \to \mathrm{SL}_n(\mathcal{O}_1)$. As mentioned in § 3.4, the set $L(\mathrm{SL})$ can be identified with $M_{\mathrm{SL}} = \{X \in \mathrm{M}_n(\mathcal{O}_1) \mid \mathrm{Tr}(X) = 0\}$, by $I + \pi X \mapsto X$. We firstly show that $\langle ., . \rangle|_{M_{\mathrm{SL}}}$ is not non-degenerate by showing that the scalar matrices lie in its radical. Let $A = aI_n \in \mathrm{M}_n(\mathcal{O}_1)$, then $p \mid n$ implies $A \in M_{\mathrm{SL}}$. Hence for any $X \in M_{\mathrm{SL}}$ we obtain $\mathrm{Tr}(AX) = a\mathrm{Tr}(X) = 0$. This implies that $\langle ., . \rangle|_{M_{\mathrm{SL}}}$ is not non-degenerate. Hence the method discussed in the last section does not work in this case.

Define an equivalence relation on $M_n(\mathcal{O}_1)$ by $A \sim B$ if there exists a scalar $x \in \mathcal{O}_1$ such that $A = xI + B$. Denote the equivalence class of A under this relation

by $[A]$ and the set of equivalence classes by $\mathfrak{L}$. The set $\mathfrak{L}$ is an abelian group under the operation $[A]+[B]=[A+B]$. For any $[A] \in \mathfrak{L}$ define $\psi_{[A]} : L(SL) \to \mathbf{C}^{\times}$ by

$$\psi_{[A]}(I+\pi X) = \psi(\mathrm{Tr}(AX)).$$

Then $\psi_{[A]}$ is well defined character of $L(SL)$ and $[A] \mapsto \psi_{[A]}$ gives an isomorphism $\mathfrak{L} \tilde{\to} \widehat{L(SL)}$. The group $\mathrm{GL}_n(\mathcal{O}_2)$ acts on $\mathfrak{L}$ by conjugation via its quotient $\mathrm{GL}_n(\mathcal{O}_1)$, and therefore on $\widehat{L(SL)}$. For $\alpha \in \mathrm{GL}_n(\mathcal{O}_2)$ and $\psi_{[A]} \in \widehat{L(SL)}$, we obtain,

$$\psi_{[A]}^{\alpha}(I+\pi X) \;=\; \psi(\mathrm{Tr}(A\kappa(\alpha)X\kappa(\alpha)^{-1})) \;=\; \psi_{\kappa(\alpha)^{-1}[A]\kappa(\alpha)}(I+\pi X).$$

Let $T_G(\psi_{[A]}) = \{\alpha \in \mathrm{GL}_n(\mathcal{O}_2) \mid \psi_{[A]}^{\alpha} = \psi_{[A]}\}$. Observe that $L(SL)$ is a subgroup of $I+\pi M_n(\mathcal{O}_2)$ and the character $\psi_A \in \widehat{K}$ restricts to $\psi_{[A]}$ on $L(SL)$. By definitions it follows that $T_G(\psi_A) = \{\alpha \in \mathrm{GL}_n(\mathcal{O}_2) \mid \psi_A^{\alpha} = \psi_A\}$ is a subgroup of $T_G(\psi_{[A]})$. Let $T_{\mathrm{SL}}(\psi_{[A]})$ be the stabilizer of $\psi_{[A]}$ in $\mathrm{SL}_n(\mathcal{O}_2)$, then $T_{\mathrm{SL}}(\psi_{[A]}) = T_G(\psi_{[A]}) \cap \mathrm{SL}_n(\mathcal{O}_2)$. We subdivide our further discussion to two cases.

4.1. **The case $T_G(\psi_A) = T_G(\psi_{[A]})$:** The condition $T_G(\psi_A) = T_G(\psi_{[A]})$ implies

$$T_{\mathrm{SL}}(\psi_{[A]}) = T_G(\psi_A) \cap \mathrm{SL}_n(\mathcal{O}_2).$$

Then define $\chi_{\psi_{[A]}} = \chi_{\psi_A}^{G}|_{T_G(\psi_A)\cap \mathrm{SL}_n(\mathcal{O}_2)}$, where $\chi_{\psi_A}^{G}$ is as obtained from Proposition 3.1. Then $\chi_{\psi_{[A]}}|_{L(SL)} = \psi_{[A]}$. This proves the existence of canonical extension, that is Proposition 2.2(a), in this case.

4.2. **The case $T_G(\psi_{[A]}) \neq T_G(\psi_A)$:** Let $T_{\mathrm{SL}}(\psi_A)$ denote the set $T_G(\psi_A) \cap \mathrm{SL}_n(\mathcal{O}_2)$. For this case, we follow the following steps to obtain the character $\chi_{\psi_{[A]}}$ of $T_{\mathrm{SL}}(\psi_{[A]})$.

Step 1: We show that the group $T_{\mathrm{SL}}(\psi_A)$ is normal in $T_{\mathrm{SL}}(\psi_{[A]})$ and the group $T_{\mathrm{SL}}(\psi_{[A]})/T_{\mathrm{SL}}(\psi_A)$ is abelian.
Step 2: There exists an abelian group $\mathcal{S}$ such that

$$T_{\mathrm{SL}}(\psi_{[A]}) = T_{\mathrm{SL}}(\psi_A)\mathcal{S},$$

and the intersection $\mathcal{S} \cap T_{\mathrm{SL}}(\psi_A)$ is trivial.
Step 3 There exists a character χ_A of $T_{\mathrm{SL}}(\psi_A)$ that is invariant in $T_{\mathrm{SL}}(\psi_{[A]})$ and hence extends to give required character $\chi_{\psi_{[A]}}$.

By definition of the action of $\mathrm{GL}_n(\mathcal{O}_2)$ on $\mathrm{M}_n(\mathcal{O}_1)$ and the set $\{[X] \mid X \in \mathrm{M}_n(\mathcal{O}_1)\}$ of equivalence classes, we obtain $T_G(\psi_{[A]}) = \{g \in \mathrm{GL}_n(\mathcal{O}_2) \mid g[A]g^{-1} = [A]\}$ and $T_G(\psi_A) = \{g \in \mathrm{GL}_n(\mathcal{O}_2) \mid gAg^{-1} = A\}$. Let $g \in T_G(\psi_{[A]})$ be such that $gAg^{-1} = A + xI$ for some $x \in \mathcal{O}_1$, and let $z \in T(\psi_A)$. Then,

$$(gzg^{-1})A(gzg^{-1})^{-1} \;=\; gzg^{-1}Agz^{-1}g^{-1} \;=\; gz(A-xI)z^{-1}g^{-1} = A.$$

This implies that $T_G(\psi_A)(T_{\mathrm{SL}}(\psi_A))$ is a normal subgroup of $T_G(\psi_{[A]})(T_{\mathrm{SL}}(\psi_{[A]}))$. Further the quotient $T_G(\psi_{[A]})/T_G(\psi_A)(T_{\mathrm{SL}}(\psi_{[A]})/T_{\mathrm{SL}}(\psi_A))$ is abelian because

$$g_1g_2A(g_1g_2)^{-1} = (g_2g_1)A(g_2g_1)^{-1}.$$

This completes the proof of Step 1.

We can assume that matrix A has all its eigenvalues in the field $\mathcal{O}_1$, for if not we can apply the argument to an extension field of F. Hence for further discussion, we shall assume that that all the eigenvalues of A lie in the field $\mathcal{O}_1$.

The assumption $T_G(\psi_{[A]}) \neq T_G(\psi_A)$ implies there exists a nonzero scalar $x \in \mathcal{O}_1$ such that A is conjugate to $xI + A$. Therefore, if a is an eigenvalue of A then so is $a + x$. We arrange the distinct eigenvalues of A in the following order,

$$a_{11}, a_{12}, \ldots, a_{1p}, a_{21}, a_{22}, \ldots, a_{2p}, \ldots, a_{r1}, a_{r2}, \ldots, a_{rp},$$

where $a_{ij} = a_{i(j-1)} + x$, $a_{ip} + x = a_{i1}$ for all $1 \leq i \leq p$,$2 \leq j \leq p$, and for $i \neq i'$, $a_{ij} - a_{i'j} \notin (x)$ (the additive space generated by x). Assume that A is in its Jordan Canonical form (see Theorem 3.5 of Singla [7]), that is

$$A = \oplus_{i=1}^{r} \oplus_{j=1}^{p} A_{ij}, \tag{4.1}$$

where each A_{ij} is further direct sum of Jordan blocks with unique eigenvalue a_{ij}.

Every element of $T_G(\psi_{[A]})$ modulo the group $T_G(\psi_A)$ just permutes the matrices A_{ij} among each other. Therefore every element of $T_G(\psi_{[A]})$ can be written as product of an element of the permutation matrix and an element of the group $T_G(\psi_A)$. The cosets of $T_{\mathrm{SL}}(\psi_A)$ in $T_{\mathrm{SL}}(\psi_{[A]})$ can be parametrized by the permutation matrices consisting of $pr \times pr$ blocks with each $(i,j)^{th}$ block of size equal to size of matrix A_{ij} and with the property that each block is either identity or zero matrix. Let $\mathcal{S}$ be the collection of the permutation matrices corresponding to the quotient $T_{\mathrm{SL}}(\psi_{[A]})/T_{\mathrm{SL}}(\psi_A)$, then $\mathcal{S}$ is an abelian group (by Step 1)) satisfying,

(1) $T_{\mathrm{SL}} = \mathcal{S}T_{\mathrm{SL}}(\psi_A)$.
(2) The intersection $\mathcal{S} \cap T_{\mathrm{SL}}(\psi_A)$ is trivial.

This proves Step 2. For step 3, first of all we briefly recall the construction of character $\chi_{\psi_A}^G$ of $T_G(\psi_A)$ that extends ψ_A, from Singla [7].

Let $s : \mathcal{O}_1^\times \to \mathcal{O}_2^\times$ be the unique multiplicative section of the natural projection map $\mathcal{O}_2^\times \to \mathcal{O}_1^\times$. By defining $s(0) = 0$, we obtain a section $s : \mathcal{O}_1 \to \mathcal{O}_2$ of the natural projection map $\mathcal{O}_2 \to \mathcal{O}_1$. By extending it entry wise, we obtain a map from $\mathrm{GL}_n(\mathcal{O}_1) \to \mathrm{GL}_n(\mathcal{O}_2)$, denoted again by s.

For a matrix A as given in (4.1), let $Z_{\mathrm{GL}_n(\mathcal{O}_2)}(s(A)) = \{g \in \mathrm{GL}_n(\mathcal{O}_2) \mid gs(A) = s(A)g\}$ and m_{ij} be the size of each matrix A_{ij}. Then

$$Z_{\mathrm{GL}_n(\mathcal{O}_2)}(s(A)) = \Pi_{i=1}^{r}\Pi_{j=1}^{p} Z_{\mathrm{GL}_{m_{ij}}(\mathcal{O}_2)}(s(A_{ij})),$$

and $T_G(\psi_A) = K.Z_{\mathrm{GL}_n(\mathcal{O}_2)}(s(A))$ (see Lemma 5.1 and Corollary 5.3 of Singla [7]).

For any $a \in \mathcal{O}_1$ define a character $\psi_a : 1+\pi\mathcal{O}_2 \to \mathbf{C}^\times$ by $\psi_a(1+\pi x) = \psi(ax)$. Since $\mathcal{O}_2^\times$ is direct product of $\mathcal{O}_1^\times$ and $1+\pi\mathcal{O}_2$, the characters ψ_a extend trivially to $\mathcal{O}_2^\times$, denote this by χ_a. Define a character χ of $Z_{\mathrm{GL}_n(\mathcal{O}_2)}(s(A)) = \Pi_{i=1}^{r}\Pi_{j=1}^{p} Z_{\mathrm{GL}_{m_{ij}}(\mathcal{O}_2)}(s(A_{ij}))$ by

$$\chi(\Pi_{i=1}^{r}\Pi_{j=1}^{p} X_{ij}) = \chi_{a_{11}}(\det(X_{11}))\ldots\chi_{a_{rp}}(\det(X_{rp})).$$

Then the character $\chi^G_{\psi_A} = \psi_A.\chi$ of $T_G(\psi_A)$, defined by $\psi_A.\chi(uv) = \psi_A(u)\chi(v)$ for all $u \in K$ and $v \in Z_{\mathrm{GL}_n(\mathcal{O}_2)}(s(A))$, satisfies $\chi^G_{\psi_A}|_K = \psi_A$. Let $\chi_A = \chi^G_{\psi_A}|_{T_{\mathrm{SL}}}(\psi_A)$. Then,

Lemma 4.1. *The one dimensional representation χ_A of $T_{\mathrm{SL}}(\psi_A)$ is fixed by $T_{\mathrm{SL}}(\psi_{[A]})$.*

Proof. To prove that χ_A is fixed by $T_{\mathrm{SL}}(\psi_A)$, it is sufficient to prove that the restriction of χ to $Z_{\mathrm{GL}_n(\mathcal{O}_2)}(s(A)) \cap \mathrm{SL}_n(\mathcal{O}_2)$ is invariant under the action of elements of the group $\mathcal{S}$, as $\psi_A|_{K\cap \mathrm{SL}_n(\mathcal{O}_2)} = \psi_{[A]}$ and $\psi_{[A]}$ is fixed by $T_{\mathrm{SL}}(\psi_{[A]})$ by definition of $T_{\mathrm{SL}}(\psi_{[A]})$.

Observe that any permutation matrix $s \in \mathcal{S}$ such that $sAs^{-1} = A + x'I$ permutes the matrices A_{ij} and $A_{i'j'}$ among each other only if both A_{ij} and $A_{i'j'}$ have the same block decomposition and $a_{ij} - a_{i'j'} \in (x')$, where (x') denotes the additive space generated by x'. Let $X \in (Z_{\mathrm{GL}_n(\mathcal{O}_2)}(s(A)) \cap \mathrm{SL}_n(\mathcal{O}_2))$. Then by the structure of $Z_{\mathrm{GL}_n(\mathcal{O}_2)}(s(A))$, we obtain that $X = \Pi_{i=1}^{r}\Pi_{j=1}^{p} X_{ij}$. Let $\det(X_{ij}) = \beta_{ij}(1+\pi\alpha_{ij})$, where $\beta_{ij} \in \mathcal{O}_1^\times$ and $1+\pi\alpha_{ij} \in 1+\pi\mathcal{O}_2$. Here we have again used the fact that $\mathcal{O}_2^\times$ is direct product of $\mathcal{O}_1^\times$ and $1+\pi\mathcal{O}_2$. Then $X \in \mathrm{SL}_n(\mathcal{O}_2)$ implies that $\Pi_{i=1}^{r}\Pi_{j=1}^{p}\beta_{ij} = 1$ and therefore $\sum_{i=1}^{r}\sum_{j=1}^{p}\alpha_{ij} = 0$. By definition of χ, we obtain

$$\chi(X) = \psi(\sum_{i=1}^{r}\sum_{j=1}^{p}(a_{ij} + (j-1)x)(\alpha_{ij}))$$

and

$$\chi(sXs^{-1}) = \psi(\sum_{i=1}^{r}\sum_{j=1}^{p}(a_{ij} + x' + (j-1)x)(\alpha_{ij})).$$

but then $\sum_{i=1}^{r}\sum_{j=1}^{p}\alpha_{ij} = 0$ implies

$$\sum_{i=1}^{r}\sum_{j=1}^{p}(a_{ij} + (j-1)x)(\alpha_{ij}) = \sum_{i=1}^{r}\sum_{j=1}^{p}(a_{ij} + x' + (j-1)x)(\alpha_{ij}).$$

Therefore $\chi(X) = \chi(sXs^{-1})$. This proves the lemma. □

Hence by Singla [7, Lemma 5.4] (also see Isaacs [3, Exercise 6.18]) χ_A extends to $T_{\mathrm{SL}}(\psi_{[A]})$. Furthermore, as $\mathcal{S}$ is an abelian group with a trivial intersection with $T_{\mathrm{SL}}(\psi_A)$, we can define extension $\chi_{\psi_{[A]}}$ canonically such that $\chi_{\psi_{[A]}}|_{\mathcal{S}}$ is trivial. This completes the proof of Proposition 2.2(a) for $\mathrm{SL}_n(\mathcal{O}_2)(p \mid n)$.

Let $Z_{\mathrm{GL}_n(\mathcal{O}_1)}([A])$ be the subgroup of $\mathrm{GL}_n(\mathcal{O}_1)$ consisting of matrices $g \in \mathrm{GL}_n(\mathcal{O}_1)$ satisfying $g[A]g^{-1} = [A]$. Parts (b) and (c) of Proposition 2.2 follow by observing,

(1) The group $T_{\mathrm{SL}}(\psi_{[A]})/L(SL)$ is isomorphic to $Z_{\mathrm{GL}_n(\mathcal{O}_1)}([A]) \cap \mathrm{SL}_n(\mathcal{O}_1)$.
(2) $|\mathrm{SL}_n(\mathcal{O}_2)/T_{\mathrm{SL}}(\psi_{[A]})| = |\mathrm{SL}_n(\mathcal{O}_1)/(Z_{\mathrm{GL}_n(\mathcal{O}_1)}([A]) \cap \mathrm{SL}_n(\mathcal{O}_1))|$.

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